

ON THE COMMUTATIVITY OF STATES IN VON NEUMANN ALGEBRAS

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ABSTRACT. The notion of commutativity of two normal states on a von Neumann algebra was defined some time ago by means of the Pedersen-Takesaki theorem. In this note we aim at generalizing this notion to an arbitrary number of states, and obtaining some results on so defined joint commutativity. Also relations between commutativity and broadcastability of states are investigated.

1. INTRODUCTION

Let φ and ω be normal faithful states on a von Neumann algebra. The celebrated Pedersen-Takesaki theorem defines commutativity of φ and ω in terms of their modular automorphism groups. If only ω is faithful then only *commuting φ with ω* is defined. We attempt to define *joint commutativity* of an arbitrary family of normal states which would generalize the one given by the Pedersen-Takesaki theorem. If the algebra in question is the full algebra $\mathbb{B}(\mathcal{H})$ of all bounded operators on a Hilbert space, then this joint commutativity amounts to the natural condition of commutativity of the density matrices of the states. Moreover, equivalence between pairwise commutativity as defined by the Pedersen-Takesaki theorem and the joint commutativity is obtained for a convex family of states.

The notion of *broadcastability* of states has become recently an object of growing interest in the field of Quantum Statistics and Quantum Information Theory (see e.g. [1, 2, 3, 4]). It turns out that it is closely related to commutativity of states. Namely, in general von Neumann algebras broadcastability implies commutativity while in atomic von Neumann algebras the two notions are equivalent.

2. PRELIMINARIES

Let \mathcal{M} be a σ -finite von Neumann algebra, let ω be a normal faithful state on \mathcal{M} , and let φ be an arbitrary normal state on \mathcal{M} . Let $\{\sigma_t^\omega : t \in \mathbb{R}\}$ be the modular automorphism group associated with

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the state ω . φ is said to *commute* with ω if

$$\varphi \circ \sigma_t^\omega = \varphi, \quad \text{for each } t \in \mathbb{R}.$$

The *centralizer* \mathcal{M}^ω of ω is defined as

$$\begin{aligned} \mathcal{M}^\omega &= \{x \in \mathcal{M} : \sigma_t^\omega(x) = x \text{ for each } t \in \mathbb{R}\} \\ &= \{x \in \mathcal{M} : \omega(xy) = \omega(yx) \text{ for each } y \in \mathcal{M}\}. \end{aligned}$$

Let ω be a normal faithful state. For a positive bounded operator a we define the normal positive functional ω_a on \mathcal{M} by

$$\omega_a(x) = \omega(a^{1/2}xa^{1/2}), \quad x \in \mathcal{M}.$$

In particular, if $a \in \mathcal{M}^\omega$ then $a^{1/2} \in \mathcal{M}^\omega$ too, and we have

$$\omega_a(x) = \omega(a^{1/2}xa^{1/2}) = \omega(ax) = \omega(xa), \quad x \in \mathcal{M}.$$

If A is a positive selfadjoint operator affiliated with \mathcal{M}^ω then we define ω_A as

$$\omega_A(x) = \lim_{\varepsilon \rightarrow 0} \omega_{A(\mathbb{1} + \varepsilon A)^{-1}}(x).$$

(The point in the above definition is that the operators $A(\mathbb{1} + \varepsilon A)^{-1}$ are bounded and $A(\mathbb{1} + \varepsilon A)^{-1} \uparrow A$ for $\varepsilon \downarrow 0$.) Assume now that A is affiliated with \mathcal{M}^ω . Then $(\mathbb{1} + A)^{-1/2}A^{1/2}(\mathbb{1} + \varepsilon A)^{-1/2}$ is in \mathcal{M}^ω , and the operators $(\mathbb{1} + A)^{-1/2}$ and $A(\mathbb{1} + \varepsilon A)^{-1}$ commute, thus for each $x \in \mathcal{M}$ we have

$$\begin{aligned} &\omega_A((\mathbb{1} + A)^{-1/2}x(\mathbb{1} + A)^{-1/2}) \\ &= \lim_{\varepsilon \rightarrow 0} \omega(A^{1/2}(\mathbb{1} + \varepsilon A)^{-1/2}(\mathbb{1} + A)^{-1/2}x(\mathbb{1} + A)^{-1/2}A^{1/2}(\mathbb{1} + \varepsilon A)^{-1/2}) \\ &= \lim_{\varepsilon \rightarrow 0} \omega((\mathbb{1} + A)^{-1/2}A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1/2}x) \\ &= \lim_{\varepsilon \rightarrow 0} \omega(A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1}x). \end{aligned}$$

The operators $A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1}$ are bounded and converge strongly as $\varepsilon \rightarrow 0$ to the bounded operator $A(\mathbb{1} + A)^{-1}$. Since

$$\|A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1}\| \leq 1,$$

we have also $A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1} \rightarrow A(\mathbb{1} + A)^{-1}$ σ -strongly, consequently, $A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1}x \rightarrow A(\mathbb{1} + A)^{-1}x$ σ -strongly, hence σ -weakly, so we have

$$\lim_{\varepsilon \rightarrow 0} \omega(A(\mathbb{1} + \varepsilon A)^{-1}(\mathbb{1} + A)^{-1}x) = \omega(A(\mathbb{1} + A)^{-1}x).$$

Thus we have obtained the formula

$$(1) \quad \omega_A((\mathbb{1} + A)^{-1/2}x(\mathbb{1} + A)^{-1/2}) = \omega(A(\mathbb{1} + A)^{-1}x), \quad x \in \mathcal{M}.$$

For a more thorough discussion of the above notions the reader is referred to [8, Sections 2.21, 4.1, 4.4, 4.8, 4.10].

In what follows we shall repeatedly make use of the Pedersen-Takesaki theorem, so for the reader's convenience we state its main points here in the setup involving states. For its full version concerning weights [8, Section 4.10] can be consulted.

Recall that for a normal state φ on \mathcal{M} the symbol $s(\varphi)$ denotes the support of φ . If ω is a normal faithful state on \mathcal{M} then $[D\varphi : D\omega]_t, t \in \mathbb{R}$, stands for the Connes cocycles (or, in other words, the Connes-Radon-Nikodym derivatives) of φ with respect to ω .

Theorem 1 (Pedersen-Takesaki). *Let ω be a faithful normal state on a von Neumann algebra \mathcal{M} , and let φ be a normal state on \mathcal{M} . The following conditions are equivalent*

- (i) $\varphi \circ \sigma_t^\omega = \varphi$ for all $t \in \mathbb{R}$ (i.e. φ commutes with ω),
- (ii) $[D\varphi : D\omega]_t \in \mathcal{M}^\omega$ for all $t \in \mathbb{R}$,
- (iii) $\{[D\varphi : D\omega]_t : t \in \mathbb{R}\}$ is a strongly continuous group of unitary elements of the algebra $s(\varphi)\mathcal{M}s(\varphi)$,
- (iv) there exists a positive selfadjoint operator A affiliated with \mathcal{M}^ω such that $\varphi = \omega_A$.

For $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, by $W^*(\mathcal{A})$ we shall denote the von Neumann algebra generated by \mathcal{A} , i.e. the smallest von Neumann algebra containing \mathcal{A} .

3. COMMUTATIVITY OF STATES

Let us begin with a simple supplement to the Pedersen-Takesaki theorem which indicates a possible generalization of the notion of commutativity of states. This result seems to be known at least for faithful states, in any case it is mentioned without proof in [7, p. 165].

Proposition 2. *Let ω be a faithful normal state on a von Neumann algebra \mathcal{M} , and let φ be a normal state on \mathcal{M} . The following conditions are equivalent*

- (i) φ commutes with ω ,
- (ii) the Connes cocycles $\{[D\varphi : D\omega]_t : t \in \mathbb{R}\}$ form a commuting family.

Proof. (i) \implies (ii). Since φ commutes with ω we have on account of Theorem 1 that $\{[D\varphi : D\omega]_t : t \in \mathbb{R}\}$ is a unitary group on the algebra $s(\varphi)\mathcal{M}s(\varphi)$, thus $[D\varphi : D\omega]_t$ and $[D\varphi : D\omega]_s$ commute for all $s, t \in \mathbb{R}$.

(ii) \implies (i). Denote $u_t = [D\varphi : D\omega]_t$. By assumption, the operators u_t commute. From the properties of the Connes cocycles (cf. [8, Section 3.1]) we have for all $t \in \mathbb{R}$

$$(2) \quad \sigma_t^\omega(s(\varphi)) = u_t^* u_t = u_t u_t^* = s(\varphi),$$

in particular, $u_t \in s(\varphi)\mathcal{M}s(\varphi)$.

Let \mathcal{R} be the von Neumann algebra generated by all u_t . From the cocycle property

$$u_{t+s} = u_t \sigma_t^\omega(u_s)$$

we obtain, taking into account equality (2),

$$u_t^* u_{t+s} = u_t^* u_t \sigma_t^\omega(u_s) = \sigma_t^\omega(s(\varphi)) \sigma_t^\omega(u_s) = \sigma_t^\omega(s(\varphi)u_s) = \sigma_t^\omega(u_s),$$

showing that $\sigma_t^\omega(u_s) \in \mathcal{R}$. It follows that $\sigma_t^\omega(\mathcal{R}) \subset \mathcal{R}$, i.e. in fact $\sigma_t^\omega(\mathcal{R}) = \mathcal{R}$. Now $(\sigma_t^\omega|_{\mathcal{R}})$ is a one-parameter group of automorphisms of \mathcal{R} such that $(\omega|_{\mathcal{R}}) \circ (\sigma_t^\omega|_{\mathcal{R}}) = \omega|_{\mathcal{R}}$, and the uniqueness of the modular automorphism group yields

$$\sigma_t^{\omega|_{\mathcal{R}}} = \sigma_t^\omega|_{\mathcal{R}}$$

(see e.g. [5, Chapter 9.2] or [9, Chapter 10.17]). But $\sigma_t^{\omega|_{\mathcal{R}}} = \text{id}_{\mathcal{R}}$ because \mathcal{R} is abelian, consequently

$$\sigma_t^\omega(u_s) = u_s,$$

showing that $u_s \in \mathcal{M}^\omega$, thus on account of Theorem 1 φ commutes with ω . \square

The notion of commutativity for two states has been defined with at least one of them being faithful, thus it is not clear how it can be generalized to a family of states which may contain also non-faithful elements, in which case the naturally-looking definition as pairwise commutativity fails. One possible attempt is presented below. As we shall see it agrees with a rather straightforward notion of commutativity for states on the algebra $\mathbb{B}(\mathcal{H})$ which can be defined simply as the commutativity of their density matrices.

Suppose that a normal state φ commutes with a faithful normal state ω . Then according to Theorem 1 we have $\varphi = \omega_A$ for some positive selfadjoint operator A affiliated with \mathcal{M}^ω . Thus A may be considered as a “density matrix” of φ with respect to ω in a way similar to the one suggested by the relation $\rho = \text{tr}_{D_\rho}$ for arbitrary normal state ρ , where D_ρ is the customary density matrix of ρ , and tr is the canonical trace on $\mathbb{B}(\mathcal{H})$. Moreover, by [8, Section 4.8] we have

$$[D\varphi : D\omega]_t = A^{it}.$$

These considerations lead to the following definition of commutativity of states.

Definition. Let Γ be an arbitrary family of normal states on a von Neumann algebra \mathcal{M} containing a faithful state. The states in Γ are said to *commute* if for arbitrary faithful state $\omega \in \Gamma$, the Connes cocycles $\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\}$ form a commuting family, i.e. the von Neumann algebra $W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\})$ is abelian.

Observe that this definition is consistent because if φ is another normal faithful state in Γ (thus, in particular, commuting with ω) then we have

$$W^*(\{[D\rho : D\varphi]_t : t \in \mathbb{R}, \rho \in \Gamma\}) = W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\}).$$

Indeed, for each $\rho \in \Gamma$ the chain rule and the formula for the inverse of the Connes cocycles (cf. [8, Sections 3.4, 3.5]) yield

$$\begin{aligned} [D\rho : D\varphi]_t &= [D\rho : D\omega]_t [D\omega : D\varphi]_t \\ &= [D\rho : D\omega]_t [D\varphi : D\omega]_t^{-1} = [D\rho : D\omega]_t [D\varphi : D\omega]_{-t}, \end{aligned}$$

since on account of Theorem 1 $\{[D\varphi : D\omega]_t : t \in \mathbb{R}\}$ is a unitary group. Consequently,

$$[D\rho : D\varphi]_t \in W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\}),$$

thus

$$W^*(\{[D\rho : D\varphi]_t : t \in \mathbb{R}, \rho \in \Gamma\}) \subset W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\}),$$

and by the same token we obtain the reverse inclusion.

For the full algebra $\mathbb{B}(\mathcal{H})$ we have

Theorem 3. *Let Γ be an arbitrary subset of all normal states on $\mathbb{B}(\mathcal{H})$ containing a faithful state. The following conditions are equivalent*

- (i) *the states in Γ commute,*
- (ii) *the density matrices of the states in Γ commute.*

Proof. Before starting a proof of the equivalence (i) \iff (ii) let us make some general remarks. For an arbitrary normal state φ on $\mathbb{B}(\mathcal{H})$ with density matrix D_φ , and the canonical trace tr we have $\varphi = \text{tr}_{D_\varphi}$, thus on account of [8, Section 4.8]

$$[D\varphi : D(\text{tr})]_t = D_\varphi^{it}.$$

Consequently, if ω is a faithful normal state, then we obtain the formula

$$\begin{aligned} (3) \quad [D\varphi : D\omega]_t &= [D\varphi : D(\text{tr})]_t [D(\text{tr}) : D\omega]_t \\ &= [D\varphi : D(\text{tr})]_t [D\omega : D(\text{tr})]_t^{-1} = D_\varphi^{it} D_\omega^{-it}. \end{aligned}$$

(i) \implies (ii). Pick a faithful state ω in Γ , and let $\rho \in \Gamma$ be arbitrary. Since ρ commutes with ω we infer on account of Theorem 1 that $\{[D\rho : D\omega]_t : t \in \mathbb{R}\}$ forms a unitary group in $s(\rho)\mathcal{M}s(\rho)$. Consequently, using formula (3) we obtain for any $s, t \in \mathbb{R}$

$$D_\rho^{it} D_\rho^{is} D_\omega^{-it} D_\omega^{-is} = D_\rho^{i(t+s)} D_\omega^{-i(t+s)} = D_\rho^{it} D_\omega^{-it} D_\rho^{is} D_\omega^{-is},$$

yielding the equality

$$D_\rho^{is} D_\omega^{-it} = D_\omega^{-it} D_\rho^{is},$$

which shows that the density matrices D_ρ and D_ω commute.

Take arbitrary $\varphi \in \Gamma$. Proposition 2 says that the Connes cocycles $[D\rho : D\omega]_t$ and $[D\varphi : D\omega]_s$ commute, so taking into account relation (3) we have

$$D_\rho^{it} D_\omega^{-it} D_\varphi^{is} D_\omega^{-is} = D_\varphi^{is} D_\omega^{-is} D_\rho^{it} D_\omega^{-it},$$

and since by virtue of the first part of the proof D_ω commutes with D_ρ and D_φ , we obtain

$$D_\rho^{it} D_\varphi^{is} = D_\varphi^{is} D_\rho^{it},$$

hence D_ρ and D_φ commute.

(ii) \implies (i). Let ω be an arbitrary faithful state in Γ . For arbitrary $\rho, \varphi \in \Gamma$ the density matrices D_ρ , D_φ and D_ω commute, which on account of relation (3) clearly gives the commutativity of the Connes cocycles, thus by virtue of Proposition 2 the commutativity of the states in Γ . \square

The theorem above can be generalized in the following way. Let \mathcal{M} be a semifinite von Neumann algebra, and let τ be a normal semifinite faithful trace on \mathcal{M} . Then we have an isometric isomorphism $\mathcal{M}_* \simeq L^1(\mathcal{M}, \tau)$ given by the formula $\mathcal{M}_* \ni \varphi \mapsto h_\varphi \in L^1(\mathcal{M}, \tau)$,

$$\varphi(x) = \tau(h_\varphi x), \quad x \in \mathcal{M}.$$

If φ is a state then h_φ is positive, and on account of [8, Section 4.8] we have

$$[D\varphi : D\tau]_t = h_\varphi^{it}, \quad t \in \mathbb{R},$$

where h_φ^{it} are unitaries in the algebra $s(\varphi)\mathcal{M}s(\varphi)$ (see [6], [10] or [11] for a more thorough account of the theory of noncommutative L^p -spaces). Now reasoning as in the proof of Theorem 3 we get

Theorem 4. *Let Γ be an arbitrary set of normal states on a semifinite von Neumann algebra \mathcal{M} , let τ be a normal semifinite faithful trace on \mathcal{M} , and assume that Γ contains a faithful state. Then the following conditions are equivalent*

- (i) *the states in Γ commute,*
- (ii) *the operators h_φ , $\varphi \in \Gamma$, commute, where by the commutativity of possibly unbounded operators h_φ and h_ψ we mean the commutativity of the families $\{h_\varphi^{it} : t \in \mathbb{R}\}$ and $\{h_\psi^{it} : t \in \mathbb{R}\}$.*

For convex Γ we have the following equivalence of commutativity and pairwise commutativity of states.

Theorem 5. *Let Γ be a convex set of normal states on a von Neumann algebra \mathcal{M} containing a faithful state. The following conditions are equivalent*

- (i) *the states in Γ commute,*
- (ii) *for arbitrary faithful state $\omega \in \Gamma$ each state $\rho \in \Gamma$ commutes with ω (“pairwise commutativity”).*

Proof. (i) \implies (ii). Obvious.

(ii) \implies (i). Take arbitrary faithful state $\omega \in \Gamma$ and arbitrary $\rho \in \Gamma$. Since ρ and ω commute we have on account of Theorem 1 that $\{[D\rho : D\omega]_t : t \in \mathbb{R}\}$ is a unitary group in the algebra $s(\rho)\mathcal{M}s(\rho)$, thus $[D\rho : D\omega]_t$ and $[D\rho : D\omega]_s$ commute for all $s, t \in \mathbb{R}$.

Let φ be another state in Γ . Then $\rho = \omega_A$, and $\varphi = \omega_B$ for some positive selfadjoint operators A, B affiliated with \mathcal{M}^ω . Put

$$a = \left(\frac{\mathbb{1} + A}{2} \right)^{-1}.$$

Then $a \in \mathcal{M}^\omega$. Consider the faithful state $\frac{\rho + \omega}{2} \in \Gamma$. We have on account of equality (1)

$$\begin{aligned} \left(\frac{\rho + \omega}{2} \right)_a(x) &= \frac{\rho + \omega}{2} \left(\left(\frac{\mathbb{1} + A}{2} \right)^{-1/2} x \left(\frac{\mathbb{1} + A}{2} \right)^{-1/2} \right) \\ &= (\rho + \omega) \left((\mathbb{1} + A)^{-1/2} x (\mathbb{1} + A)^{-1/2} \right) \\ &= \rho \left((\mathbb{1} + A)^{-1/2} x (\mathbb{1} + A)^{-1/2} \right) + \omega \left((\mathbb{1} + A)^{-1/2} x (\mathbb{1} + A)^{-1/2} \right) \\ &= \omega_A \left((\mathbb{1} + A)^{-1/2} x (\mathbb{1} + A)^{-1/2} \right) + \omega \left((\mathbb{1} + A)^{-1} x \right) \\ &= \omega \left(A (\mathbb{1} + A)^{-1} x \right) + \omega \left((\mathbb{1} + A)^{-1} x \right) \\ &= \omega \left(\left(A (\mathbb{1} + A)^{-1} + (\mathbb{1} + A)^{-1} \right) x \right) = \omega(x). \end{aligned}$$

Thus we have obtained that

$$\omega = \left(\frac{\rho + \omega}{2} \right)_a,$$

which on account of [8, Section 4.8] yields the equality

$$\left[D\omega : D \frac{\rho + \omega}{2} \right]_t = a^{it}.$$

Analogously, putting

$$b = \left(\frac{\mathbb{1} + B}{2} \right)^{-1},$$

we obtain that

$$\omega = \left(\frac{\varphi + \omega}{2} \right)_b,$$

hence

$$\left[D\omega : D \frac{\varphi + \omega}{2} \right]_t = b^{it}.$$

By the chain rule and the formula for the inverse of the Connes cocycles we get

$$\begin{aligned} \left[D \frac{\varphi + \omega}{2} : D \frac{\rho + \omega}{2} \right]_t &= \left[D \frac{\varphi + \omega}{2} : D\omega \right]_t \left[D\omega : D \frac{\rho + \omega}{2} \right]_t \\ &= \left[D\omega : D \frac{\varphi + \omega}{2} \right]_t^{-1} \left[D\omega : D \frac{\rho + \omega}{2} \right]_t = b^{-it} a^{it}. \end{aligned}$$

Since the states $\frac{\varphi + \omega}{2}$ and $\frac{\rho + \omega}{2}$ commute we infer, again by Theorem 1, that

$$\left\{ \left[D \frac{\varphi + \omega}{2} : D \frac{\rho + \omega}{2} \right]_t = b^{-it} a^{it} : t \in \mathbb{R} \right\}$$

is a unitary group, which yields for all $s, t \in \mathbb{R}$ the equality

$$b^{-is} b^{-it} a^{is} a^{it} = b^{-i(s+t)} a^{i(s+t)} = b^{-is} a^{is} b^{-it} a^{it},$$

and thus

$$b^{-it} a^{is} = a^{is} b^{-it}.$$

Consequently, the unitary groups $\{a^{it} : t \in \mathbb{R}\}$ and $\{b^{it} : t \in \mathbb{R}\}$ commute, which yields that a and b commute. It follows that A^{it} and B^{is} commute for all $s, t \in \mathbb{R}$. Since by virtue of [8, Section 4.8] we have

$$[D\rho : D\omega]_t = A^{it}, \quad [D\varphi : D\omega]_s = B^{is},$$

condition (i) follows. \square

Finally, let us say a few words about connections between commutativity and broadcastability of states. Recall that a family of normal states Γ on a von Neumann algebra \mathcal{M} is said to be *broadcastable* if there is a normal unital completely positive map $K: \mathcal{M} \overline{\otimes} \mathcal{M} \rightarrow \mathcal{M}$ (called a *channel*) such that for each $\rho \in \Gamma$ we have

$$\rho(K(x \otimes \mathbb{1})) = \rho(K(\mathbb{1} \otimes x)) = \rho(x), \quad x \in \mathcal{M}.$$

Assume now that Γ contains a faithful state ω . Then from [4, Theorem 12] it follows that the states in Γ commute. If \mathcal{M} is atomic then we have also the reverse implication. Namely, the von Neumann algebra $\mathcal{R} = W^*(\{[D\rho : D\omega]_t : t \in \mathbb{R}, \rho \in \Gamma\})$ is abelian, and as was shown in the proof of Proposition 2, for the modular automorphism group (σ_t^ω) we have

$$\sigma_t^\omega([D\rho : D\omega]_s) = [D\rho : D\omega]_s,$$

which yields the equality

$$\sigma_t^\omega(\mathcal{R}) = \mathcal{R}.$$

(As a matter of fact this equality is valid in the general case irrespective of the abelianess of the algebra \mathcal{R} .) Thus there exists a normal faithful conditional expectation from \mathcal{M} onto \mathcal{R} which implies that this algebra is atomic (since \mathcal{M} was such). Now from [4, Theorem 12] it follows that Γ is broadcastable.

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